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SELF-CONVERSE AND ORIENTED GRAPHS AMONG THE THIRD PARTS OF NEARLY COMPLETE DIGRAPHS

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An arc decomposition of a digraph H into t isomorphic parts is generalized so that either a remainder R or a surplus S, both of the numerically smallest possible size, are allowed. The sets of such nearly t^{th} parts are defined to be the floor class $\lfloor H/t \rfloor_R$ $(=(H-R)/t)$ and the ceiling class $\lceil H/t \rceil_S$ $(=(H+S)/t)$, respectively. We restrict ourselves to the case of nearly third parts of $H=\mathcal{D}K_n$, the complete digraph, with $t=3$. Then $|R|=0=|S|$ if $n \not\equiv 2 \pmod 3$, else $|R|=2$ and $|S|=1$. The existence of nearly third parts which are oriented graphs and/or self-converse digraphs is settled in the affirmative for all or most n's. Moreover, it is proved that floor classes with distinct R's ($|R|=2$) can have a common member. The corresponding result on the nearly third parts of the complete 2-fold graph ${}^2K_n$ is deduced. Furthermore, $\bigcap_R \lfloor {}^2K_n/3 \rfloor_R \neq \emptyset$ also if $|R|=2$.

Introduction

Let $\mathcal{D}K_n$ denote the complete digraph on n vertices (without loops and multiple arcs). The complete λ-fold graph on n vertices without loops, where λ is the multiplicity of each edge, is denoted by ${}^\lambda K_n$. In this paper we shall consider extremal packings and coverings of $\mathcal{D}K_n$ or ${}^2K_n$ by $t=3$ isomorphic parts.

Note that in Harary et al. [5] it is proved that $\mathcal{D}K_n$ can be decomposed into t isomorphic digraphs if and only if the number of arcs, $n(n-1)$, is a multiple of t. Our aim is to generalize the special case $t=3$ in two ways. The first basic way consists in applying the floor and ceiling notions as defined in Skupień [8] so as to allow for all orders n to be dealt with. The second way is by proving the existence of a third or nearly third part having certain additional properties, for instance to be a self-converse digraph (possibly with 2-dicycle) and/or an oriented graph.

Before we state our results, let us agree about terminology.

Following Iverson (1962, see Graham et al. [3]) we use the names *floor* and *ceiling* of a real number x and corresponding symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ to stand for the integer part of x and the least integer $\geq x$, respectively. Recall also that, given positive integers m and t, the symbol $m \bmod t$ stands for the remainder on dividing m by t.

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In what follows, $G=(V,E)$ (where $|V|=n$ and $|E|=m$) stands for a digraph or multigraph on n vertices and m *lines*, each of which is an arc or edge depending on what G stands for. In case G is a digraph, we use the notation $E=A(G)$, the set of arcs; else $E=E(G)$ is the set of edges.

Let $R=R_{G,t}$ and $S=S_{G,t}$ be a set and possibly a multiset, respectively, of lines of G and $G\cup\overline{G}$, respectively, such that $|R|=m \bmod t$ and $|S|=(t-|R|) \bmod t$. Hence $|S|=|R|=0$ if $t\,|\,m$, else $|R|+|S|=t$. The digraphs [multigraphs] induced by R and S are denoted by $\langle R\rangle$ and $\langle S\rangle$, respectively.

By a *decomposition* of a (labelled) G we mean a family of line-disjoint substructures (subdigraphs or submultigraphs) of G which include all lines of G. Those substructures are called *∂-parts* (or *parts of the decomposition*). We write $t\,|\,G$ if there exists a decomposition of G into t mutually isomorphic ∂-parts. Moreover, the isomorphism class of those isomorphic ∂-parts is called a t^{th} *part* of G.

G/t stands for the class of t^{th} parts of G provided that $t\,|\,m$. In general, given an R as described above, let $\lfloor G/t\rfloor_R := (G-R)/t$. Then $\lfloor G/t\rfloor_R$ is called the *floor class* (with the *remainder* R). Similarly, $\lceil G/t\rceil_S := (G+S)/t$ is called the *ceiling class* (with the *surplus* S).

In what follows $t=3$. Then a remainder R and surplus S have cardinalities $|R|=0=|S|$ if $n\not\equiv 2 \pmod 3$, else $|R|=2$ and $|S|=1$. Therefore S, if nonempty, comprises a copy, a, of an arc. Hence, if $3\nmid n(n-1)$, we shall write a in place of S. One can see that there are five possibilities for the remainder R if R is nonempty.

In some cases, however, we shall find common members of floor classes with distinct remainders.

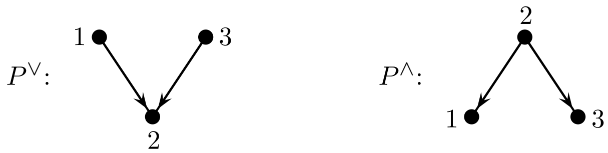

Fig. 1. The gutter and the roof

Recall that a digraph without any 2-dicycle is called the oriented graph. A directed tree on three vertices such that two is the indegree [outdegree] of a vertex is called the *gutter* [*roof*] (see Fig. 1).

We are going to prove the following results.

Theorem 1. *For each $n\geq 3$ and any admissible and self-converse R, the floor class $\lfloor \mathcal{D}K_n/3\rfloor_R$ contains a self-converse oriented graph unless either $n=3$ or possibly $n=8$ and $\langle R\rangle=\vec{P}_3$.*

It is reported in [6] that a computer has not found such a member in case $n=8$ and $\langle R\rangle=\vec{P}_3$.

Theorem 2. *If* $k \in \mathbb{N}$ *and* $n = 3k-1 \neq 5$ *then* $S = \{a\}$ *and the ceiling class* $\lceil \mathcal{D}K_n/3 \rceil_a$ *contains a self-converse oriented graph.*

However, if $n = 5$, a self-converse member with 2-dicycle exists, which is included in the next result.

Theorem 3. *For each* $n \geq 3$ *and any admissible R and S where R is self-converse, each class* $\lfloor \mathcal{D}K_n/3 \rfloor_R$ *and* $\lceil \mathcal{D}K_n/3 \rceil_S$ *contains a member with 2-dicycle. Moreover, such a member can be assumed to be self-converse unless* $n = 5$ *and* $R = 2a$*. Furthermore, if* $R \neq \emptyset$ *and* $R \neq 2a$ *if* $n = 5$*, all floor classes with a fixed n have a common member which is a self-converse digraph with 2-dicycle.*

Corollary. *For each* $n \geq 3$ *and any admissible R and S where R is self-converse, each class* $\lfloor \mathcal{D}K_n/3 \rfloor_R$ *and* $\lceil \mathcal{D}K_n/3 \rceil_S$ *contains a self-converse digraph.* ∎

Theorem 4. *For each* $n \geq 3$ *and any admissible R and S, both the floor class* $\lfloor \mathcal{D}K_n/3 \rfloor_R$ *and ceiling class* $\lceil \mathcal{D}K_n/3 \rceil_S$ *contain an oriented graph.*

Theorem 5. *For any* $n \geq 3$ *and any admissible R and S, each of classes* $\lfloor {}^2K_n/3 \rfloor_R$ *and* $\lceil {}^2K_n/3 \rceil_S$ *contains a simple graph with an even number of vertices of any fixed odd degree and with automorphism which fixes neither any edge nor any vertex of odd degree.*

0. Preliminaries

We use standard notation and terminology of graph theory. Notions not defined here can be found in [4], [2], [1].

In this paper, the symbol $\cup$ when applied to digraphs or multigraphs stands for the *vertex-disjoint* union. The symbol $+$ when applied to digraphs [multigraphs] stands for the *line-disjoint* union. Moreover, if E' is a set of (possibly copies of) lines, and $E' \cap E = \emptyset$ then $G + E'$ denotes the spanning supermultidigraph [spanning supermultigraph] of G with the line set $E \cup E'$. Likewise, $G - E'$ denotes the spanning digraph [spanning multigraph] obtained from G by removing a set E' of lines. We write $G \pm E' = G \pm l$ if $E' = \{l\}$, $l = e$ (edge) or $l = a$ (arc). Given an integer λ, the union of λ disjoint copies of G is denoted by λG, λl being the set of λ disjoint lines.

The ordered pair (v_1, v_2) of vertices v_1 and v_2 (or the symbol $v_1 \to v_2$) denotes the arc which goes from v_1 to v_2.

Given a multigraph G, let $\mathcal{D}G$ denote a digraph obtained from G by bijectively replacing each edge with a pair of opposite arcs without changing the endvertices.

Let $[v_1, \ldots, v_n]$ denote an *undirected* v_1*–*v_n *path*, P_n, whose consecutive vertices are $v_1, \ldots, v_n$ and are assumed pairwise distinct. Then only for $n \geq 3$, $[v_1, \ldots, v_n, v_1] := P_n + v_n v_1 = C_n$, an *undirected n-cycle*. Furthermore, the symbol $\mathcal{D}([v_1, \ldots, v_n])$ stands for $\mathcal{D}P_n$ if $v_1 \neq v_n$, else for $\mathcal{D}C_{n-1}$ (with $n > 3$). Each of the following digraphs on $n \geq 2$ vertices is an oriented graph unless it is a 2-dicycle $\vec{C}_2$:

(i) a *dipath* $\vec{P}_n$, $\vec{P}_n = [v_1, \ldots, v_n]^{\rightarrow}$ where $v_1 \neq v_n$;

(ii) a *dicycle* $\vec{C}_n = \vec{P}_n + (v_n, v_1) = [v_1, \ldots, v_n, v_1]^{\rightarrow}$;

(iii) a *matching* $2\vec{P}_2$ (actually, a digraph induced by a 2-matching).

If ϕ is a permutation of $V(G)$, let ϕ' be the induced permutation which acts on lines, e.g., $\phi'(i,j) = (\phi i, \phi j)$. Then ϕG stands for the image of G under ϕ, $\phi G := (V, \phi'[E])$ where $\phi'[E]$ is the image of the line set E under ϕ'. Given a self-converse digraph G on n vertices, we use the symbol φ $(= \varphi_G = \varphi_n)$ to denote a *conversing permutation*, that is, a permutation of $V(G)$ such that φG is the converse digraph of G. Similarly, let G be a part of a cyclic decomposition of a structure H into t parts. Unlike Bosák [2] we assume that there is a permutation γ $(= \gamma_G = \gamma_n)$ of $V(H)$ which *generates* the cyclic decomposition, i.e., the image of G under the j^{th} iterate $(\gamma)^j$ of γ is the $(j+1)^{\text{th}}$ part $(\gamma)^j G$ of the decomposition, where $j = 0, \ldots, t-1$; $(\gamma)^0 = id$. Call γ to be a *placing permutation* (or *placement-generating permutation*) for G. More generally, the symbol ψ_i $(= \psi_{n,i})$ stands for the i^{th} *placing function* for G, ψ_i being a permutation of $V(H)$ such that $\psi_i G$ is the i^{th} part of a certain decomposition, $i = 1, \ldots, t$; $\psi_1 = id$. Then the sequence $\psi_1 (= id), \ldots, \psi_t$ is called a *placing system* for G. Such a placing system will be represented by the subsequence $\psi_2, \ldots, \psi_t$ or (if possible) by a placing permutation γ.

There are two non-self-converse digraphs of size two each, namely,
$P^{\vee} = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_3, v_2)\})$, a gutter,
$P^{\wedge} = (\{v_1, v_2, v_3\}, \{(v_2, v_1), (v_2, v_3)\})$, a roof.
The gutter and the roof are mutually converse (Fig. 1). Note that

$$\mathcal{D}K_3/3 = \{\vec{C}_2,\ P^{\vee},\ P^{\wedge}\}$$

and $\gamma = (123)$ is a placing permutation for each member.

1. Proof 1

Lemma 1.1. *For $n = 4, \ldots, 9$ and any admissible and self-converse remainder R as well as for $n = 14$ and $\langle R \rangle = \vec{P}_3$, the floor class $\lfloor \mathcal{D}K_n/3 \rfloor_R$ contains a self-converse oriented graph unless possibly $n = 8$ and $\langle R \rangle = \vec{P}_3$.*

Proof. A required element of the floor class in question will be denoted by D_n or D_n^x. We assume that the symbol D_n^x with a superscript x is used only if $R \neq \emptyset$, i.e., $n = 3k+2 \geq 5$, $k \in \mathbb{N}$ and $|R| = 2$. Moreover, x therein is the order of $\langle R \rangle$; namely, $x = 2, 3, 4$ if $\langle R \rangle = \vec{C}_2, \vec{P}_3, 2\vec{P}_2$, respectively. Assume that vertices of D_n (and D_n^x)

are $v_1, \ldots, v_n$ with the exception of $n=9$ in which case $V(D_9)=\{u_1, \ldots, u_9\}$. In what follows, especially in figures and in descriptions of digraphs or corresponding permutations, we shall use the letter i to denote the vertex v_i (or u_i if $n=9$).

Consider the following six digraphs together with their associated permutations (a conversing one and a placing system).

$A(D_4)=\{1\to 2,3;\, 3\to 4;\, 4\to 2\},\ \varphi_4=(12)(34),\ \gamma_4=(123).$

$A(D_5^2)=\{1\to 2,4;\, 2\to 3;\, 3\to 1;\, 4\to 5;\, 5\to 2\},\ \varphi_5^2=(12)(45),\ \gamma_5^2=(124),$
$R=\{3\to 5;\, 5\to 3\}.$

$A(D_5^3)=\{1\to 3,4;\, 2\to 1;\, 3\to 2;\, 4\to 5;\, 5\to 2\},\ \varphi_5^3=\varphi_5^2,$
$\psi_{5,2}^3=(12)(35),\ \psi_{5,3}^3=(14)(23),\ R=A([v_3,v_5,v_4]^{\to}).$

$A(D_5^4)=A(D_5^2),\ \varphi_5^4=\varphi_5^2,\ \psi_{5,2}^4=(13425),\ \psi_{5,3}^4=(142),\ R=\{2\to 4;\, 5\to 3\}.$

$A(D_6)=\{1\to 3,4,5;\, 2\to 1;\, 3\to 2,5;\, 4\to 2;\, 5\to 6;\, 6\to 2,4\},$
$\varphi_6=(12)(34)(56),\ \gamma_6=(145)(236).$

$A(D_9)=\{1\to 2,6,8,9;\, 2\to 3,4,6,9;\, 3\to 1,5,7;\, 4\to 1;\, 5\to 1,2;$
$6\to 3,7;\, 7\to 1,2,9;\, 8\to 2,7;\, 9\to 3,5,8\},\ \varphi_9=(12)(56)(79),$
$\gamma_9=(147)(258)(369).$

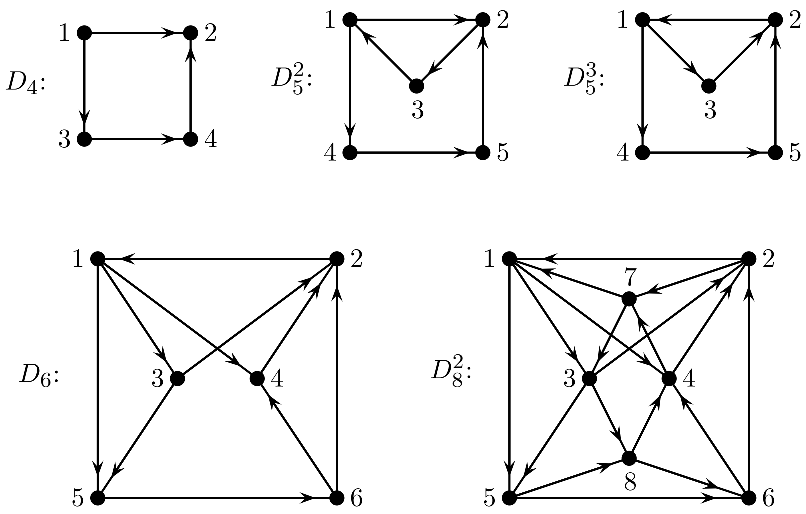

Fig. 2. Self-converse oriented graphs for $n=4$, 5, 6, 8

Note that all the above digraphs are depicted in Fig. 2 and Fig. 3. The fifth digraph in Fig. 2, D_8^2, identical with D_8^4, is described below. A graphical representation of each of those digraphs is chosen so that the symmetry of the vertex set about vertical line is a conversing permutation of the digraph.

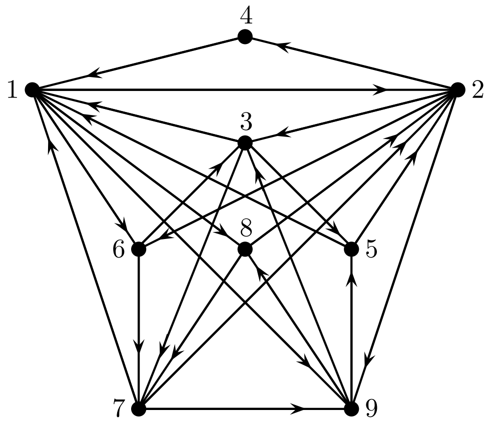

Fig. 3. Self-converse oriented graph D_9

It can be seen that each of the six digraphs presented above as well as its associated permutations have really required properties.

Three of the digraphs are used in the following definitions of digraphs D_7, D_8^2, D_8^4 and D_{14}^3. Given permutations α and β of disjoint sets, their common extension to the union of the sets is denoted by (α,β). Clearly, $[v]=K_1$. Define

$D_7=([v_7]\cup D_6)+\{(v_2,v_7),(v_4,v_7),(v_7,v_1),(v_7,v_3)\}$,

$\varphi_7=(\varphi_6,id)$, $\gamma_7=(\gamma_6,id)$ where $id\colon v_7\mapsto v_7$;

$D_8^2=([v_8]\cup D_7)+\{(v_3,v_8),(v_5,v_8),(v_8,v_4),(v_8,v_6)\}$,

$\varphi_8^2=(\varphi_7,id)$, $\gamma_8^2=(\gamma_7,id)$; $R=\{7\to 8; 8\to 7\}$;

$D_8^4=D_8^2$, $\varphi_8^4=\varphi_8^2$, $\psi_{8,2}^4=(184)(263)$, $\psi_{8,3}^4=(16827435)$,

$R=\{4\to 8; 7\to 5\}$.

For $x=3$, define

$$D_{14}^x=(D_5^x\cup D_9)+\{(v_i,u_1),(v_i,u_5),(v_i,u_9),(u_2,v_i),(u_6,v_i),(u_7,v_i)\colon v_i\in V(D_5^x), i=1,\dots,5, u_j\in V(D_9), j=1,\dots,9\},$$

$$v_{k+5}:=u_k,\ k=1,\dots,9,\ \varphi_{14}^x=(\varphi_5^x,\varphi_9),\ \psi_{14,i}^x=(\psi_{5,i}^x,(\gamma_9)^{i-1}),$$

$$i=2,3,\ R=R^x=A([v_3,v_5,v_4]^{\rightarrow}).$$

One can easily check that the above constructions give what is really required. ∎

Note that each construction at the end of the above proof takes the advantage of the structure of the involved digraph D_m ($m=6$, 9), its placing permutation γ_m and the conversing one, φ_m, cf. the last paragraph in the following proof of Theorem 1.

Proof of Theorem 1. For $n=3$, $\vec{P_3}$, the only subdigraph of $\mathcal{D}K_3$ which is a self-converse oriented graph of size two, is not a third part of $\mathcal{D}K_3$. For $n>3$ we proceed by induction on the order n. Lemma 1.1 can be viewed as the first step in our proof. Assume now that our theorem is true for a fixed admissible n ($n>3$; $n\neq 8$ if $\langle R\rangle=\vec{P}_3$). Let D_n stand for a corresponding element of the floor class $\lfloor \mathcal{D}K_n/3\rfloor_R$, let vertices of D_n be denoted by $u_j\,(j=1,\ldots,n)$, let φ_n be a conversing permutation for D_n and let either γ_n or $\psi_{n,i}$, $i=2,3$, stand for a placing system for D_n. We are going to show that Theorem is true for n replaced by $n+6$.

Using the digraph D_6 (Fig. 2), with vertices v_i, define D_{n+6} to be the following digraph.

$$D_{n+6}=(D_n\cup D_6)+\{(u_j,v_1),(u_j,v_3),(v_2,u_j),(v_4,u_j)\colon u_j\in V(D_n),\\ j=1,\ldots,n\},\ u_{n+i}:=v_i,\,i=1,\ldots,6.$$

Then, on omitting a possible superscript x, the associated permutations are $\varphi_{n+6}=(\varphi_n,\varphi_6)$ for each n, $\gamma_{n+6}=(\gamma_n,\gamma_6)$ if $3\,|\,n(n-1)$, else $\psi_{n+6,i}=(\psi_{n,i},(\gamma_6)^{i-1})$, $i=1,2,3$; the remainder remains unchanged.

The construction of D_{n+6} above takes the advantage of the structure of D_6. Namely, all arcs of D_{n+6} going from D_n to D_6 are all those from D_n to a transversal, v_1 and v_3, of the cycles in the permutation γ_6 (which generates a triple placement of D_6 in $\mathcal{D}K_6$). Another transversal, v_2 and v_4, covers all arcs of D_{n+6} going from D_6 to D_n. Since $D_n\in\lfloor \mathcal{D}K_n/3\rfloor_R$, it follows that $D_{n+6}\in\lfloor \mathcal{D}K_{n+6}/3\rfloor_R$ with the unchanged remainder R. Since, moreover, those two transversals can be matched into 2-cycles of φ_6 (φ_6 being a conversing permutation for D_6), and D_n is self-converse, so is D_{n+6}. It is clear that D_{n+6} is an oriented graph. ∎

Note that the idea behind the constructions of D^x_{14} and D_7 involving D_9 and D_6, respectively, in the proof of Lemma 1.1 is the same as that used above in the construction involving D_6.

Proposition 1.2. *Assume $n=3k+2\geq 5$ and $\langle R\rangle=\vec{C}_2,\ \vec{P}_3,\ 2\vec{P}_2$. Consider three corresponding floor classes $\lfloor \mathcal{D}K_n/3\rfloor_R$ with the exception of $n=8$ and $\langle R\rangle=\vec{P}_3$ when only two remaining classes are to be considered. Then, for fixed n, each of classes contains a self-converse oriented graph $\hat{D}^x_n$ (where x is the order of $\langle R\rangle$, $x=2,3,4$) such that all three/two digraphs $\hat{D}^x_n$ have the same underlying graph, the digraphs with $x=2,4$ being isomorphic.*

Proof. Using notation from the proof of Lemma 1.1, define $\hat{D}^x_n=D^x_n$ for each admissible x and $n=5,8,14$ ($x\neq 3$ if $n=8$) where, for $n=14$, remainders $R=R^x$ being the same as those for $n=5$. Note that requirements imposed in Proposition are met. Proceed by induction on n as in proof of Theorem 1. Apply the same construction involving the digraph D_6 in order to go from $\hat{D}^x_n$ to $\hat{D}^x_{n+6}$. Use our digraphs for $n=5,14$ as starting structures. ∎

It can be seen that floor classes involved in Proposition 1.2 are distinct because they have different cardinalities (for $n=5$ cardinalities are 3, 2 for $x=2$, 4, respectively, which was found using a computer, see [6]).

2. Proof 2

Lemma 2.1. *For $n=2$, 11 and S comprising a copy a of an arc, the ceiling class $\lceil \mathcal{D}K_n/3\rceil_a$ contains a self-converse oriented graph.*

Proof. For $n=2$, $\vec{P_2} \in \lceil \mathcal{D}([u_1,u_2])/3\rceil_a$. If $a \sim (u_1,u_2)$ and $\vec{P_2} = [u_1,u_2]^{\rightarrow}$ then $\varphi_2 = \gamma_2 = (12)$. To define D_{11}, use D_9 as in the proof of Lemma 1.1.

$D_{11} = (D_9 \cup [u_{10},u_{11}]^{\rightarrow}) + \{(u_i,u_1),(u_i,u_5),(u_i,u_9),(u_2,u_i),(u_6,u_i),(u_7,u_i),$
$i=10,11\}$, $\varphi_{11} = (\varphi_9,(10\ 11)$, $\gamma_{11} = (\gamma_9,(10\ 11))$.

One can check that D_{11} is a required digraph for $n=11$ and $a \sim (u_{10},u_{11})$. ∎

Proof of Theorem 2. We proceed by induction on k. For $k=1,4$, required digraphs D_{3k-1} ($D_2 := \vec{P_2}$) and the associated permutations φ_{3k-1} and γ_{3k-1} are presented in the proof of Lemma 2.1. Assume that our theorem is true for a fixed admissible k (where $n=3k-1$, $k\in\mathbb{N}$ and $k\neq 2$). Let D_n stand for a corresponding element of the ceiling class $\lceil \mathcal{D}K_n/3\rceil_a$ with the associated permutations φ_n and γ_n The construction of the digraph D_{n+6} and its associated permutation φ_{n+6} follows by the same method as in the proof of Theorem 1, but $\gamma_{n+6} = (\gamma_n,\gamma_6)$. ∎

3. Proof 3

Lemma 3.1. *For a fixed $n=3,4$, or $n=5$ and either $\langle R\rangle = \vec{C_2}$ or $\langle R\rangle = \vec{P_3}$, or $n=8$ and any self-converse R with $|R|=2$, all floor classes contain a common self-converse member with 2-dicycle.*

Proof. A required element of the floor class will be denoted by F_n and its vertices $v_1,\ldots,v_n$. Letters φ, γ, ψ will be used as described before to denote new associated permutations as well. A possible superscript x will stand for the order of the digraph $\langle R\rangle$ if $R\neq\emptyset$.

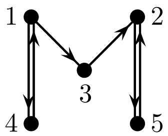

Fig. 4. The digraph F_5

Consider the following definitions.

$F_3 = \mathcal{D}([v_1,v_2]) \cup [v_3]$, $\varphi_3 = id$, $\gamma_3 = (123)$.

$F_4 = \mathcal{D}([v_1, v_2]) \cup \mathcal{D}([v_3, v_4])$, $\varphi_4 = id$, $\gamma_4 = (123)$.
$F_5 = [v_4, v_1, v_3, v_2, v_5]^{\rightarrow} + \{(v_1, v_4), (v_5, v_2)\}$, $\varphi_5 = (12)(45)$, see Fig. 4;
for $x = 2$, $\gamma_5^2 = (124)$ and then $R = \{(v_3, v_5), (v_5, v_3)\}$;
for $x = 3$, $\psi_{5,2}^3 = (1452)$, $\psi_{5,3}^3 = (125)$ and then $R = A([v_5, v_3, v_4]^{\rightarrow})$.
$A(F_8) = \{1 \to 2, 4, 7; 2 \to 3, 5; 3 \to 2, 7; 4 \to 1, 6, 8; 5 \to 3; 6 \to 1, 5, 8;$
$7 \to 3, 5; 8 \to 2, 4\}$, $\varphi_8 = (12)(34)(56)(78)$, see Fig. 5;
for $x = 2$, $\gamma_8^2 = (145)(236)$ and then $R = \{(v_7, v_8), (v_8, v_7)\}$;
for $x = 3$, $\psi_{8,2}^3 = (145)(236)$, $\psi_{8,3}^3 = (1854)(263)$ and then
$R = A([v_5, v_7, v_8]^{\rightarrow})$;
for $x = 4$, $\psi_{8,2}^4 = (145)(236)$, $\psi_{8,3}^4 = (1854)(2763)$ and then
$R = \{(v_5, v_6), (v_7, v_8)\}$.
It is easy to check that the above constructions give what is required. ∎

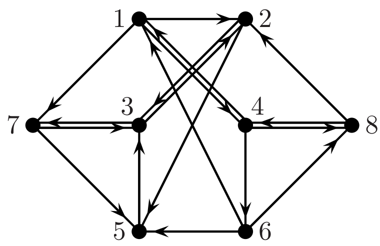

Fig. 5. The digraph F_8

Lemma 3.2. *For $n = 5$ and $S = \{a\}$ where a is copy of an arc, the ceiling class $\lceil \mathcal{D}K_5/3 \rceil_a$ contains a self-converse digraph $\tilde{F}_5$ with a dicycle $\vec{C}_2$.*

Proof. Put
$\tilde{F}_5 = (\mathcal{D}([v_1, v_2]) \cup \mathcal{D}([v_4, v_3, v_5]) + (v_4, v_5)$, $\tilde{\varphi}_5 = (45)$, $\tilde{\gamma}_5 = (123)(45)$,
$a \sim (v_4, v_5)$.
One can see that these definitions are correct. ∎

Lemma 3.3. *For $n = 5$ and $R = 2a$ the floor class $\lfloor \mathcal{D}K_5/3 \rfloor_{2a}$ contains a non-self-converse digraph with 2-dicycle.*

Proof. One can check that the following digraph $\tilde{D}_5$ meets our requirements.
$A(\tilde{D}_5) = \{1 \to 3; 2 \to 1, 4; 3 \to 1; 4 \to 5; 5 \to 2\}$,
$\tilde{\psi}_{5,2} = (142)$, $\tilde{\psi}_{5,3} = (1325)$, $R = \{(v_3, v_5), (v_4, v_2)\}$. ∎

Proof of Theorem 3. We proceed by induction on n. Lemmas 3.1–3.3 are the first step in our proof. Assume now that Theorem is true for a fixed $n \geq 3$ and $n \neq 5$

in case of the floor. Let F_n stand for a corresponding element of the floor class $\lfloor\mathcal{D}K_n/3\rfloor_R$, vertices of F_n being u_j, $j=1,\ldots,n$. We now prove that the floor part of Theorem is true for n replaced by $n+3$. Use our F_3 and define

$$F_{n+3}=(F_n\cup F_3)+\{(u_j,v_3),(v_3,u_j): u_j\in V(F_n), j=1,\ldots,n\},$$
$$u_{n+i}:=v_i,\ i=1,2,3.$$

Then, on omitting a possible superscript x, the associated permutations are $\varphi_{n+3}=(\varphi_n,\varphi_3)$ for each n, $\gamma_{n+3}=(\gamma_n,\gamma_3)$ if $3\,|\,n(n-1)$, else $\psi_{n+3,i}=(\psi_{n,i},(\gamma_3)^{i-1})$, $i=1,2,3$.

Same constructions can be used if F_n is a corresponding element of the ceiling class $\lceil\mathcal{D}K_n/3\rceil_S$, $n=2+3k$, $k\in\mathbb{N}$, with the exception that $\gamma_{n+3}=(\gamma_n,\gamma_3)$ for each n. ∎

4. Proof 4

Lemma 4.1. *If $n=5$ and R is non-self-converse ($\langle R\rangle=P^\vee, P^\wedge$; a gutter or a roof) then the floor class $\lfloor\mathcal{D}K_5/3\rfloor_R$ contains an oriented graph.*

Proof. A would-be element of the floor class is denoted by D_5^ρ where ρ $(=\vee,\wedge)$ represents the shape of the remainder R.

Consider the following digraphs (which are oriented graphs and mutually converse, see Fig. 6).

$A(D_5^\vee)=\{1\to 2,3,4;\ 2\to 3;\ 5\to 2,4\},\ R=\{3\to 5;\ 4\to 5\}.$

$A(D_5^\wedge)=\{2\to 1,5;\ 3\to 1,2;\ 4\to 1,5\},\ R=\{5\to 3,4\}.$

Note that they have the same placing system $\psi_{5,2}=(12)(35)$, $\psi_{5,3}=(14)(23)$. ∎

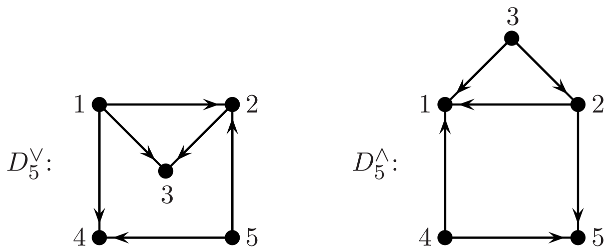

Fig. 6. The floor third parts for $n=5$

Proof of Theorem 4. We proceed by induction on n. Lemmas 1.1 and 4.1 can be viewed as the first step in our proof. Assume now that Theorem is true for a fixed n. Let D_n stand for a corresponding element of the floor class $\lfloor\mathcal{D}K_n/3\rfloor_R$, vertices of D_n being u_j, $j=1,\ldots,n$. Let $D_3=P^{\vee}$ (or $P^{\wedge}$) and $\gamma_3=(123)$. We prove that Theorem is true for n replaced by $n+3$. Define

$$D_{n+3}=(D_n\cup D_3)+\{(u_j,v_1),(v_3,u_j): u_j\in V(D_n), j=1,\ldots,n\},$$
$$u_{n+i}:=v_i,\ i=1,2,3.$$

Then the associated permutations are $\gamma_{n+3}=(\gamma_n,\gamma_3)$ if $3\mid n(n-1)$, else $\psi_{n+3,i}=(\psi_{n,i},(\gamma_3)^{i-1})$, $i=1,2,3$.

The same construction with $n=2$ and $D_n=\vec{P}_2=[u_1,u_2]^{\rightarrow}$ gives a required element of the ceiling class $\lceil\mathcal{D}K_5/3\rceil_S$. This together with Theorem 2 completes the proof. ∎

5. Proof 5 and Concluding Remarks

Proof of Theorem 5. Underlying graphs of digraphs used in the above proof of Theorem 4 meet requirements of Theorem 5, which is easy to see. ∎

Besides the case $n=8$ and R being a 3-dipath in Theorem 1, Theorems 1 and 3 leave one more case unsettled.

Conjecture. *For $n=3k+2\geq 5$ and R being a non-self-converse remainder for $t=3$ (i.e., $\langle R\rangle$ being a gutter or a roof, $|R|=2$), there is no self-converse digraph in $\lfloor\mathcal{D}K_n/3\rfloor_R$.*

Conjecture is proved for $n=5$ in [7] and is confirmed by a computer for $n=8$, see [6].

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